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Higher Order Leptonic Weak Interactions

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Higher order leptonic weak interactions as mediated by a charged intermediate boson are studied on the basis of summing the leading terms in perturbation theory. An important question is whether it is consistent to assume that such a procedure is meaningful and leads to finite results. This is indeed the case for the uncrossed ladder graphs, as shown by Feinberg and Pais. Here an attempt is made to study all higher order graphs. The regularization procedure adopted is the ξ-limiting formalism of Lee and Yang, and the leading terms taken into account are those which, for a given power of the coupling constant, contain the highest power of ξ^{-1} and also the highest possible power of $\ln \xi$ consistent with this maximum power of ξ^{-1}. The sum is determined by a comparison with the electrodynamics of vector mesons on the assumption that it is meaningful to sum the leading terms in both theories. It is then found that, for four-fermion allowed processes at energies much below 300 BeV, no inconsistencies are found and the conventional g^2 should be replaced by $\frac{5}{8}g^2$.

1. INTRODUCTION

RECENTLY, Feinberg and Pais[1] pioneered the study of higher order effects in weak interactions. In particular, they have given reasons for the need of better understanding in this direction. For a problem as complicated as this, in order to make any progress at all it is necessary to make a number of assumptions, such as the existence of a charged intermediate boson $W^{\pm}$ which mediates all leptonic weak interactions and the negligence of strong and electromagnetic interactions. To get an orientation, Feinberg and Pais[1] have studied the set of uncrossed ladder graphs under the mathematical assumption that this set can be summed to get finite results. They have discussed, among other topics, the so-called allowed processes[1] for both "low energies" (energies much below 300 BeV.) and "high energies" (energies much above 300 BeV.). The low-energy result has been put on a somewhat firmer basis,[2] still under all the assumptions mentioned above. In this paper, we restrict ourselves entirely to energies much below 300 BeV.

It has been repeatedly emphasized that there is so far no basis to think that the uncrossed ladder graphs are more important than other graphs. It is therefore the purpose of this paper to free ourselves of the restriction to ladder graphs. Since the low-energy result of the Feinberg-Pais theory is obtainable by summing the leading terms of the uncrossed ladder graphs, we attempt to sum in this paper the leading terms of all possible graphs. Since the leading term must be independent of the external momenta[2] for any graph that has only four external Fermion lines and cannot be made disconnected by removing one internal line, without any calculation we know that the result must be of the following form. For the so-called allowed leptonic processes,[1] the factor

$$-i(\delta_{\mu\nu}+m^{-2}q_\mu q_\nu)/(q^2+m^2) \tag{1.1}$$

in the expression for the matrix element should be replaced, when higher order effects are taken into account, by

$$-i[(\delta_{\mu\nu}+m^{-2}q_\mu q_\nu)/(q^2+m^2)-\eta m^{-2}\delta_{\mu\nu}]. \tag{1.2}$$

Here m denotes the mass of W, g is the W-lepton coupling constant in the Lagrangian density[3]

$$L_{\text{int}}=ig[\psi_e^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_\nu\Phi_\lambda^*+\psi_\mu^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_{\nu'}\Phi_\lambda^*]+\text{Hermitian conjugate}, \tag{1.3}$$

and η is a dimensionless real number to be determined by detailed calculation from the theory under consideration. The low-energy result of the Feinberg-Pais theory may be summarized by giving the following value for η:

$$\eta=\tfrac{1}{4}. \tag{1.4}$$

On the other hand, the considerations in this paper lead instead to

$$\eta=\tfrac{3}{8}. \tag{1.5}$$

Unlike the Feinberg-Pais theory, here we do not have any iterative scheme of computing the leading terms. Instead, (1.5) is obtained by a comparison with the electrodynamics of vector mesons, studied in detail by Lee and Yang.[4] In order to make this comparison, it is essential to use the ξ-limiting process of Lee and Yang[4] as a method of regularization also in the theory

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[1] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).

[2] Y. Pwu and T. T. Wu, Phys. Rev. **133**, B778 (1964).

[3] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960). Their notation is used in (1.3).

[4] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); and T. D. Lee, *ibid.* **128**, 899 (1962).

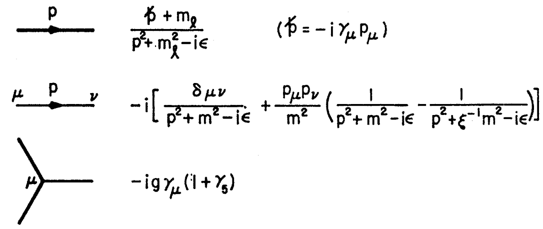

FIG. 1. Feynman rules for the W-lepton system.

of weak interactions, as was done previously[2]. The method of calculating the leading terms for leptonic weak interactions is discussed in Sec. 2, that for vector meson electrodynamics in Sec. 3, and the two theories are compared in Sec. 4.

2. LEADING TERMS FOR LEPTONIC WEAK INTERACTIONS

The interaction Lagrangian density (1.3) leads to the Feynman rules shown in Fig. 1. Since there is no positively charged lepton, it is convenient, for the boson propagator, to draw the arrow in the direction of W^-. This does not affect the Feynman rules. For simplicity, we shall call this theory A.

Consider the lowest order graph for W-W scattering, as shown in Fig. 2. Even with the ξ-limiting process,

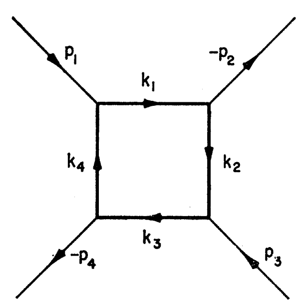

FIG. 2. Lowest order graph for W-W scattering via leptons.

this graph is logarithmically divergent. When the fermion masses are neglected and an additional momentum cutoff Ω is introduced because of this logarithmic divergence, then the leading term for this graph is

$$-4i(3\pi^2)^{-1}g^4(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\nu\rho}\delta_{\sigma\mu}-2\delta_{\mu\rho}\delta_{\nu\sigma})\ln(\Omega/Q), \quad (2.1)$$

where Q is the square root of the order of magnitude of two of the quadratic invariants formed from the four boson momenta. In (2.1) μ and ρ are the indices for the incoming W^- lines, while ν and σ are the indices for the outgoing W^- lines. Renormalization calls for the subtraction of the same term with Q set equal to the W mass. Therefore, when all the W momenta are of the order of magnitude[2] $\Lambda=\xi^{-1/2}m$, the leading term of this graph after renormalization is

$$-2i(3\pi^2)^{-1}g^4(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\nu\rho}\delta_{\sigma\mu}-2\delta_{\mu\rho}\delta_{\nu\sigma})\ln\xi. \quad (2.2)$$

This graph is discussed in some more detail in Appendix A.

In connection with the Feinberg-Pais theory, a set of simplified Feynman rules has previously been used[2] to get the leading term. Because of the result of the last paragraph, this set must be supplemented by (2.2) multiplied by 4, since there are four possible fermion loops according to (1.3), as shown in Fig. 3. The resulting set of simplified Feynman rules is shown in Fig. 4. The general applicability of these rules is discussed in Appendix B.

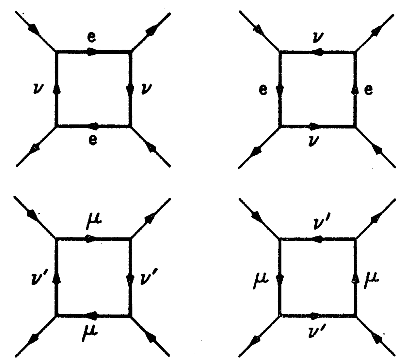

FIG. 3. Four possibilities for W-W scattering via leptons in the lowest order.

For any given power of g, the order of magnitude of the leading term may be found from these simplified Feynman rules. In view of the logarithmic factor in (2.2), it is desirable to have as many W-W scattering vertices as possible. We shall restrict our consideration to graphs with no external W line and four external lepton lines: one μ line, one e line, one ν line and one ν' line. The orders of magnitude are then as follows. The coefficient of g^2 is obtained from the usual second order theory, and is finite in the limit $\xi\to 0$. For higher powers of g, attention is confined to these graphs that cannot be made disconnected by removing one internal line, since otherwise the graph merely gives a correction to the W-lepton vertex. Under this restriction, it is

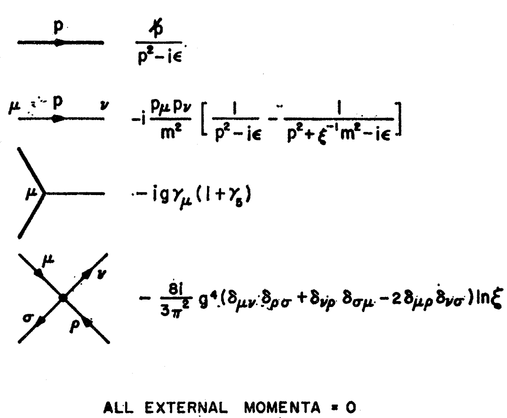

FIG. 4. Simplified Feynman rules for the W-lepton system.

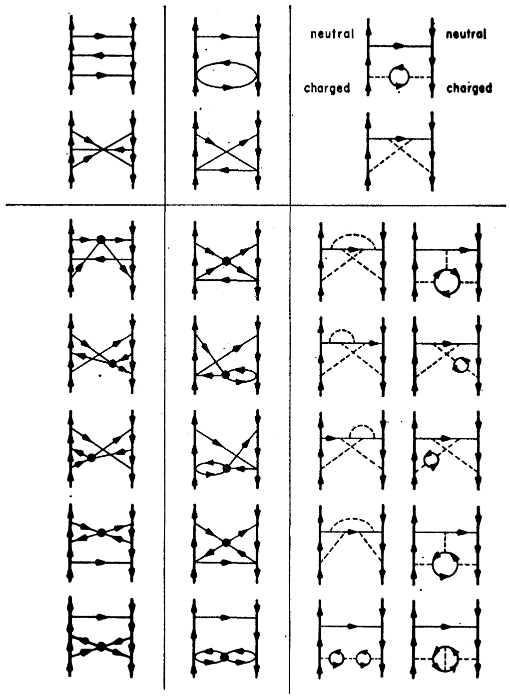

FIG. 5. Comparison of graphs for theory A and theory B. Here in the first column are shown the W-lepton graphs that contribute to the leading terms for $n=1$, 2. The resulting graphs after contraction of the neutral lepton lines as given in Fig. 8 are shown in the second column. In the third column are shown the graphs from theory B, which after contraction of the photon lines also lead to those given in the second column.

found that the coefficient of g^{4n+2} is of the order of magnitude $\xi^{-2n}(\ln\xi)^{n-1}$, while the coefficient of g^{4n+4} is of the order of magnitude $\xi^{-2n-1}(\ln\xi)^{n-1}$, where n is a positive integer. We therefore neglect all terms in-

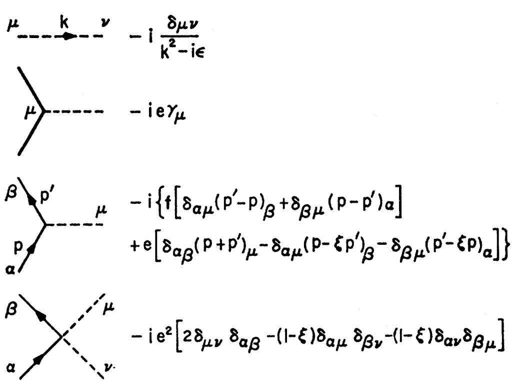

FIG. 6. Additional Feynman rules for the W-lepton-photon system.

volving g^{4n+4}. Consequently, besides the second order contribution, we need to sum a series of the following form

$$G(g,\xi)=\sum_{n=1}^{\infty} a_n g^{4n+2}\xi^{-2n}(\ln\xi)^{n-1}, \qquad (2.3)$$

where a_n is a sequence of numbers that depend only on n. The graphs that contribute to a_1 and a_2 are shown in the first column of Fig. 5. Note that each of these and high order graphs contains exactly six W-lepton vertices, three on each lepton line.

It is the purpose of the next two sections to find the sum (2.3) by a comparison with the electrodynamics of vector mesons.

3. ELECTRODYNAMICS OF VECTOR MESONS

Using the ξ-limiting formalism, Lee[4] has obtained a number of electromagnetic properties of the W. However, besides the quadruple moment, the consideration is really concerned with radiative corrections to the lowest order weak interactions. For simplicity, we shall use the name theory B for the theory where weak interactions are taken into account only to the lowest order but radiative corrections to all orders are retained. The Feynman rules for theory B are shown in Fig. 6, in addition to those already given in Fig. 1. In Fig. 6, we have used the notation

$$f=e\kappa. \qquad (3.1)$$

Therefore, there are three coupling constants, namely e, f, and g.

We proceed to obtain the simplified Feynman rules for this case, again for a graph which cannot be made disconnected by removing one internal line. First, for a vertex with two internal W lines and one internal photon line, it is sufficient to retain the part proportional to f; and the W-photon four vertex can be dropped altogether. Next, all internal photon lines that are connected to four internal W lines can be shrunk. The simplified Feynman rule for the resulting W vertex is[4]

$$if^2(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma}-2\delta_{\mu\rho}\delta_{\nu\sigma}), \qquad (3.2)$$

provided again that μ and ρ are the indices for the incoming W^- lines, while ν and σ are the indices for the outgoing W^- lines.

It remains to consider those internal photon lines that are connected to charged lepton lines. For those connected to four lepton lines, no simplification is possible. For those connected to two lepton lines and two internal W lines, contraction of the term for the W-photon vertex from Fig. 6 with the two W momenta p_α and p_β' gives

$$if[p_\mu(kp')-p_\mu'(kp)]. \qquad (3.3)$$

By the conservation of current, a term proportional to k_μ cannot affect the final answer. Therefore, (3.3) can be replaced by either

$$-ifp_\mu k^2 \quad \text{or} \quad -ifp_\mu' k^2. \qquad (3.4)$$

When (3.4) is used, the photon line can again be shrunk, and the Feynman rule for the resulting vertex with two W lines and two charged lepton lines is either

$$-ef\not{p} \quad \text{or} \quad -ef\not{p}'. \qquad (3.5)$$

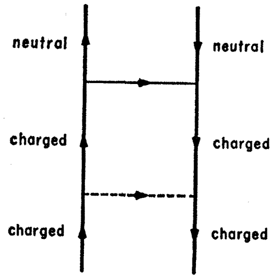

FIG. 7. Lowest radiative correction to lepton-lepton scattering.

So far as the leading terms are concerned, the situation with radiative corrections to a four-lepton process allowed by second-order weak interaction is thus as follows within the framework of theory B. Here the leading term means the dominating term as $\xi \to 0$ for a given *total* power of e and f. Besides the original second-order graph, it is sufficient to consider only graphs that cannot be disconnected by removing one internal line, since other graphs can only lead to wave function renormalization and coupling constant renormalization. The simplified Feynman graph contains, besides any number of W lines, lepton lines, and four W vertices, two W-lepton vertices and either one internal photon line which is connected to two pairs of charged lepton lines or two vertices each joining two W lines and two charged lepton lines. The simplified Feynman rules of Fig. 4 can be used[5] except for the four W vertices, which is replaced by (3.2). In addition, the usual Feynman rules for the photon propagator and the lepton-photon vertex are used, while the rule for the vertex with two W lines and two charged lepton lines is given by (3.5), after the removal of two p_μ factors from the adjacent W lines.

We note that, with these simplified Feynman rules, the result for the matrix element is always of second order in g and of second order in e, but can be of any even power in f. Consider first those graphs with an internal photon line which cannot be shrunk as described above. The graph of lowest order is shown in Fig. 7. Indeed, any higher order graph of this class can differ from that of Fig. 7 only in the addition of a self-energy insertion in the internal W line. By power counting, the contribution from the graph of Fig. 7 is of the order of $\ln\xi$. Therefore, the contributions from the higher order graphs in this class are all negligible because of the mass renormalization for the W. Accordingly, the series that needs to be summed is of the following form:

$$F(f,\xi)=g^2e^2\left[F_0(\xi)+\sum_{n=1}^{\infty} b_n f^{2n}\xi^{-2n}\right], \tag{3.6}$$

where F_0 is the contribution from the graph of Fig. 7, while all the other terms come from graphs without any internal photon line that is not shrunk. Like a_n, b_n is also a sequence of numbers that depend only on n.

[5] See, however, the end of Sec. 4.

4. COMPARISON OF THE TWO THEORIES

It is the main theme of the present paper that there is a simple relation between the two sequences of numbers a_n and b_n, and that this relation enables us to evaluate G in the limit $\xi \to 0$.

In order to get this relation, consider a graph that contributes to G. Along each of the two lepton lines, there are three W-lepton vertices, which lead to a factor $(1+\gamma_5)^3=4(1+\gamma_5)$. Next, consider the lines joined to the two charged external lepton lines. As shown diagrammatically in Fig. 8, the adjacent neutral lepton lines may also be shrunk. After this is carried out, the graphs of theory A are formally identical to those of theory B, discussed in the last section. Detailed comparison of graphs is shown in Fig. 5 in a few simple cases.

It is therefore only necessary to take proper account of the difference in the two sets of Feynman rules. If we formally identify f with g^2, then it is only necessary to compare (2.2) for theory A with (3.2) for theory B. Therefore,

$$a_n=16b_n[-8/(3\pi^2)]^{n-1}. \tag{4.1}$$

Accordingly,

$$F(g^2,\xi)=g^2e^2[F_0(\xi)+g^{-2}(-\tfrac{1}{2}\ln\xi')(3\pi^2)^{-1}G(g,\xi')], \tag{4.2}$$

where ξ and ξ' are related by

$$\xi^2=-8(3\pi^2)^{-1}\xi'^2\ln\xi'. \tag{4.3}$$

We now assume that both F and G approach finite limits as $\xi \to 0$. Then it follows from (4.2) that

$$\lim_{\xi\to 0}G(g,\xi)=-g^2 6\pi^2 \lim_{\xi\to 0}(-\ln\xi)^{-1}F_0(\xi). \tag{4.4}$$

The evaluation of $F_0(\xi)$ from the graph of Fig. 7 is straightforward, provided that, to avoid divergence, the simplified Feynman rule for the W line is not used.

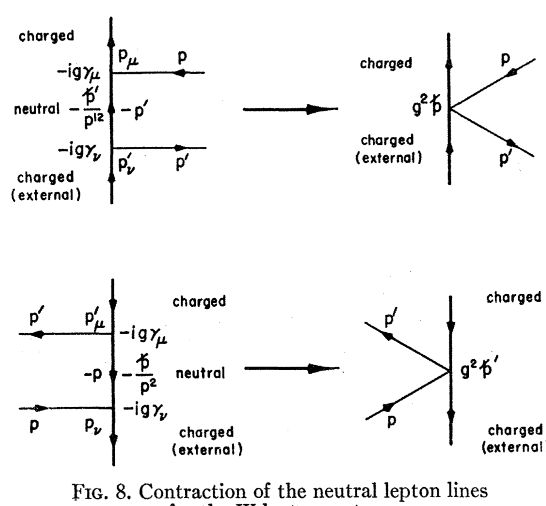

FIG. 8. Contraction of the neutral lepton lines for the W-lepton system.

The result is that

$$\lim_{\xi\to 0}G(g,\xi) = -i\tfrac{3}{8}g^2m^{-2}[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\mu(1+\gamma_5)]^{(2)}. \quad (4.5)$$

The final result (1.5) is an immediate consequence of (4.5).

5. ASSUMPTIONS INVOLVED

A number of assumptions have been made in order to obtain the final result (1.5). The following three pertain particularly to the present consideration:

(1) It has been assumed that a meaningful result can be obtained by summing the leading terms for theory A. This is a pure assumption devoid of any basis. Furthermore, here the term "leading terms" refers specifically to terms that contain not only the highest power of ξ^{-1} but in addition also the highest power of $\ln\xi$. So far as the authors know, such a situation has not been encountered previously in any physical problem of summing the leading terms. One of the consequences of this complication is discussed in the next section.

(2) The same assumption has also been made for theory B with $f\neq 0$. This is also a pure assumption. It can perhaps be said that at the present stage of understanding about higher order weak interactions it is difficult to avoid assumptions of this type. So far as theory B is concerned, there is no complication due to $\ln\xi$.

(3) Even after the removal of the logarithmic divergence due to the lowest order graph for W-W scattering by using (2.2) instead of (2.1), the resulting graphs for lepton-lepton scattering in theory A still contain numerous divergences. These divergences are also present in theory B. In order to obtain the finite coefficients a_n and b_n, some program of removing the divergences is necessary. We have assumed that the two renormalization programs for theory A and theory B are sufficiently similar so that the formal equality (4.1) is not disturbed.

6. DISCUSSIONS

The question may be raised as to what contributions are to be expected from the neglected terms. In particular, in Sec. 2, we have explicitly omitted a series of the form

$$\sum_{n=1}^{\infty} c_n g^{4n+4}\xi^{-2n-1}(\ln\xi)^{n-1}, \quad (6.1)$$

where again c_n is a sequence of numbers that depend only on n. This series is of the form

$$g^2(\ln\xi)^{-3/2} \text{ function of } (g^2\xi^{-2}\ln\xi). \quad (6.2)$$

If this has a finite limit as $\xi\to 0$, then this limit is again of the order of g^2. Therefore, because we have retained only terms with the highest powers of ξ^{-1} and the highest powers of $\ln\xi$, the sum of the next highest order terms, if finite, is of the same order of magnitude as that of the leading terms. This is in marked contrast with, for example, the case of the many body problem of hard sphere bosons, discussed critically by Lee, Huang, and Yang.[6] Here all that we have found is the first term of an infinite series of terms of comparable magnitude, while the next term is given by (6.2) which we do not know how to evaluate. It can only be hoped that this infinite series converges so very rapidly that the first term gives a reasonable approximation to the sum.

When the same procedure is applied to a forbidden process,[1] we do get the answer zero. This zero comes from dividing the matrix element for Møller scattering by the factor $(\ln\xi)^{1/2}$. However, in view of the discussion in the last paragraph, this does not imply that the forbidden processes are really forbidden to the order g^2. Again, it seems difficult to compute the sum of the next order terms.

We conclude with the remark that, if W couples to $e-\nu$ and $\mu-\nu'$ with two different coupling constants, then (1.5) is replaced by

$$\eta=\tfrac{3}{4}(x^2+x^{-2})^{-1}, \quad (6.3)$$

where x is the ratio of the two coupling constants.

ACKNOWLEDGMENTS

One of us (T. T. Wu) is greatly indebted to Professor C. N. Yang for numerous lectures on the electrodynamics of vector mesons in general and very enlightening discussions on the present problem in particular. He would also like to thank Professor A. Pais for introducing him to the subject, and Professor J. R. Oppenheimer for his hospitality at the Institute for Advanced Study.

APPENDIX A

We evaluate the lowest order graph for W-W scattering, as shown in Fig. 2; we are only interested in the contraction with the four external momenta:

$$I=-g^4(2\pi)^{-4}\int d^4k\, p_{1\mu}p_{2\nu}p_{3\rho}p_{4\sigma} \times \mathrm{Tr}\frac{1}{k_1}\gamma_\nu(1+\gamma_5)\frac{1}{k_2}\gamma_\rho(1+\gamma_5) \times\frac{1}{k_3}\gamma_\sigma(1+\gamma_5)\frac{1}{k_4}\gamma_\mu(1+\gamma_5). \quad (A1)$$

Since the external momenta are set equal to zero in the

[6] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957); K. Huang, T. D. Lee, and C. N. Yang, *Symposium on the Many Body Problem held at Stevens Institute of Technology in 1957* (Interscience Publishers, Inc., New York, 1963).

simplified Feynman rules used to calculate the leading terms of fermion scattering graphs, we are permitted to use Euclidean metric. Thus we change the problem slightly and ask for

$$I=-ig^4(2\pi)^{-4}8p_{1\mu}p_{3\rho}J_{\mu\rho}, \quad (A2)$$

where

$$J_{\mu\rho}=\int d_4k p_{2\nu}p_{4\sigma}\,\mathrm{Tr}\frac{1}{k_1}\gamma_\nu\frac{1}{k_2}\gamma_\rho\frac{1}{k_3}\gamma_\sigma\frac{1}{k_4}\gamma_\mu, \quad (A3)$$

and the integration is over the Euclidean 4-space.

To integrate (A3), we introduce the cut-off factor $e^{-\beta k^2}$ to avoid divergence. We have

$$J_{\mu\rho}=4[J_{\mu\rho}^{(13)}-J_{\mu\rho}^{(23)}-J_{\mu\rho}^{(14)}+J_{\mu\rho}^{(24)}], \quad (A4)$$

where

$$J_{\mu\rho}^{(ij)}=\frac{1}{4}\int d_4k e^{-\beta k^2}\,\mathrm{Tr}\frac{k_i}{k_i^2}\gamma_\rho\frac{k_j}{k_j^2}\gamma_\mu$$

$$=\int d_4k e^{-\beta k^2}\frac{k_{i\rho}k_{j\mu}+k_{i\mu}k_{j\rho}-(k_ik_j)\delta_{\mu\rho}}{k_i^2k_j^2}. \quad (A5)$$

Define:

$$\mathcal{J}_{\mu\nu}=\int d_4k e^{-\beta k^2}\frac{(k+P)_\mu(k+P')_\nu}{(k+P)^2(k+P')^2}. \quad (A6)$$

If

$$P=k_i-k \quad\text{and}\quad P'=k_j-k,$$

then

$$J_{\mu\rho}^{(ij)}=\mathcal{J}_{\mu\rho}+\mathcal{J}_{\rho\mu}-\delta_{\mu\rho}\mathcal{J}_{\sigma\sigma}. \quad (A7)$$

We evaluate $\mathcal{J}_{\mu\nu}$ by introducing Feynman parameters. After carrying out a tedious calculation, we can write, when β is small,

$$\mathcal{J}_{\mu\nu}=\pi^2\{\tfrac{1}{2}\delta_{\mu\nu}[(2\beta)^{-1}-R(P-P')^2-\tfrac{1}{6}(P^2+P'^2)] +(R-\tfrac{1}{6})P_\mu'P_\nu+(R+\tfrac{1}{3})P_\mu P_\nu' -R(P_\mu P_\nu+P_\mu'P_\nu')\}, \quad (A8)$$

where

$$R=\int_0^\infty dx\int_0^\infty dx'\frac{x(x'+1)}{(x+x'+1)^4}\exp\left[-\beta\frac{xx'}{x+x'+1}(P-P)^2\right]$$

$$=\tfrac{1}{6}\left[\ln\frac{1}{\beta(P-P')^2}-\gamma\right]+\tfrac{5}{36}. \quad (A9)$$

Substituting (A8) into (A7), we have

$$J_{\mu\rho}^{(ij)}=\pi^2\{-\delta_{\mu\nu}(2\beta)^{-1}+\delta_{\mu\nu}[2R(P-P')^2 +\tfrac{1}{6}(P^2-PP'+P'^2)]+(2R+\tfrac{1}{6})(P_\mu P_\nu'+P_\mu'P_\nu) -2R(P_\mu P_\nu+P_\mu'P_\nu')\}. \quad (A10)$$

Let $k=k_1$. With this choice, we have $P=0$ and $P'=p_2+p_3$ in $J_{\mu\rho}^{(13)}$, $P=p_2$ and $P'=p_2+p_3$ in $J_{\mu\rho}^{(23)}$, $P=0$ and $P'=-p_1$ in $J_{\mu\rho}^{(14)}$ and $P=p_2$ and $P'=-p_1$ in $J_{\mu\rho}^{(24)}$. Therefore, from (A4) and (A2), we have

$$I=-i2g^4\pi^{-2}\{2R^{(13)}[(p_1p_2)(p_3p_4)-(p_1p_3)(p_2p_4)] +2R^{(24)}[(p_2p_3)(p_1p_4)-(p_1p_3)(p_2p_4)] +\tfrac{1}{6}[(p_1p_2)(p_3p_4)+(p_2p_3)(p_1p_4)-(p_1p_3)(p_2p_4)]\}$$

$$=-i\tfrac{2}{3}g^4\pi^{-2}\left\{\left[\ln\frac{1}{\beta(p_2+p_3)^2}-\gamma+\frac{4}{3}\right][(p_1p_2)(p_3p_4) -(p_1p_3)(p_2p_4)]+\left[\ln\frac{1}{\beta(p_1+p_2)^2}-\gamma+\frac{4}{3}\right] \times[(p_2p_3)(p_1p_4)-(p_1p_3)(p_2p_4)]\right\} -i\tfrac{1}{6}g^4\pi^{-2}[(p_1p_3)(p_2p_4)], \quad (A11)$$

whose leading term is identical to (2.1) contracted with the four boson momenta, provided that $\beta=\Omega^{-2}$.

APPENDIX B

Consider a connected graph G with only W-lepton vertices. Let n be the number of vertices, f the number of internal lepton (fermion) lines, F the number of external lepton lines, w the number of internal W lines, W the number of external W lines, and L the number of loops. These numbers are related by

$$n=2w+W=f+F/2 \quad (B1)$$

and

$$L=n/2+1-(F+W)/2. \quad (B2)$$

When the simplified Feynman rules are used, there is the danger of introducing infrared divergences that are not present originally. In this Appendix, we discuss the conditions under which this extraneous divergence does not occur. For this purpose, it is sufficient to consider graphs that cannot be made disconnected by removing one internal line.

Let G' be a connected subgraph of G, and let n', f', w', W', and L' have the obvious meanings. Let F_1' be the number of lepton lines which are internal for G but external for G', and let W_1' be the corresponding number for W lines. Also let F_2' be the number of lepton lines external to both G and G', and W_2' again the corresponding number for W lines. Clearly

$$F'=F_1'+F_2'$$

and

$$W'=W_1'+W_2'.$$

Suppose that every momentum in G' is zero. Then the phase space volume is zero to the power $4L$ if $G=G'$ and $4(L'+F_1'+W_1'-1)$ otherwise. On the other hand, with the simplified Feynman rules, the number of powers in the denominator is $F_1'+f'$. There is no extraneous infrared divergence if the former number is always larger than the latter number for every choice of the subgraph G'.

First, assume $G=G'$. Here the condition is simply

$$4L-f>0.$$

By (B1) and (B2), this is equivalent to

$$W+F/2<2(L+1). \quad \text{(B3)}$$

Note that $L>0$. Next let $G\neq G'$. Since G cannot be made disconnected by removing one internal line, we have

$$F_1'+W_1'>1. \quad \text{(B4)}$$

In this case, the condition is that

$$F_1'+f'<4(L'+F_1'+W_1'-1).$$

Again by (B1) and (B2), this is equivalent to

$$W_2'+F_2'/2<2(L'-1)+3W_1'+\tfrac{5}{2}F_1'. \quad \text{(B5)}$$

By (B4), the right-hand side of (B5) is at least 3.

The simplified Feynman rules can be used to obtain the leading term for a graph G that cannot be made disconnected by removing one internal line provided that (B3) is satisfied for G and that (B5) is satisfied for every connected subgraph $G'\neq G$. In particular, these conditions are always satisfied if G has four external lepton lines and no external W line.

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Monte Carlo Calculations Related to the Analysis of Ultrahigh-Energy Nucleon-Nucleon Interactions*

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The Monte Carlo method is applied to an investigation of angular-distribution parameters currently in use in studies of ultrahigh-energy nuclear interactions. Specifically, center-of-mass system properties of meson showers having a total multiplicity of 16 and corresponding to nucleon-nucleon collisions at 3×10^{12} eV are calculated using the Monte Carlo method and employing input information based largely on experimentally determined average properties of ultrahigh-energy nuclear interactions. The resulting meson showers conserve energy and momentum, the angular distributions on the average possess forward-backward symmetry in the center-of-mass system, and the produced particles of each shower possess only those correlations introduced by energy-momentum conservation. After making an exact transformation of the showers to the laboratory system, some conventional analysis procedures are carried out with the resulting Monte Carlo jets using parameters calculated from the angular distributions of the charged particles. The results give insight into the sensitivity of the parameters to approximations used in the interpretation of the parameters and indicate how well, on the average as well as for individual jets, a parameter $Y(\theta)$ is a measure of the physical quantity y which the parameter is expected to represent. The parameters are studied using the means and standard deviations of their $\log[Y(\theta)/y]$ distributions. The Castagnoli energy is found, rather independently of the details of the Monte Carlo calculations, to be an overestimate of the energy of a jet by an average factor of about 1.8 (antilogarithm of the mean), with a factor 2.3 (antilogarithm of the standard deviation) defining the approximate 68% confidence interval for statistical fluctuations about this average factor in the case of individual jets. Seven other parameters are examined and are found to be generally more sensitive to the details of the Monte Carlo calculations. The factor defining the approximate 68% confidence interval for statistical fluctuations of individual $Y(\theta)/y$ values about the average factor ranges from 1.5 to 3.1 for these other parameters. An application of the $(x-\langle x\rangle)/\sigma$ analysis of the Krakow-Warsaw group indicates that, at least for the jet models considered, fluctuations will not result in spurious two-center effects being indicated by the analysis.

I. INTRODUCTION

In many studies of ultrahigh-energy interactions, experimenters must of necessity deduce most of the properties of the interactions from only the angular distributions of the emitted shower particles. In order to properly evaluate the significance of information so obtained, it is rather important to know the effects of the approximations used as well as the effects of statistical fluctuations. Partly in an attempt to ascertain these effects, we have carried out some common analysis procedures employing the angular distributions of meson showers (jets) which have been calculated by Monte Carlo methods from well-defined input information, much of which is based on experimental data from nuclear interactions at energies around 10^{12} eV. The results of such analysis procedures are compared with the information which the analyses are expected to provide.

The Monte Carlo jets may be utilized in another

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